Suggested Solution to Exercise 1

1. Find the magnitude and direction of the following vectors:

   (a) $(-1, 3)$,

   (b) $(0, -4, 9)$,

   (c) $(1, 0, 5, -3)$.

   Solution.

   (a) magnitude $= \sqrt{(-1)^2 + 3^2} = \sqrt{10}$; direction $= \frac{1}{\sqrt{10}}(-1, 3)$

   (b) magnitude $= \sqrt{0^2 + (-4)^2 + 9^2} = \sqrt{97}$; direction $= \frac{1}{\sqrt{97}}(0, -4, 9)$

   (c) magnitude $= \sqrt{1^2 + 0^2 + 5^2 + (-3)^2} = \sqrt{35}$; direction $= \frac{1}{\sqrt{35}}(1, 0, 5, -3)$

2. Find all vectors that are perpendicular to the given ones:

   (a) $(-1, 2)$,

   (b) $(2, 4, -1)$,

   (c) $(-1, 1, 1, -4)$.

   Solution.

   (a) Let (a, b) be any vector in $\mathbb{R}^2$, then (a, b) is perpendicular to $(-1, 2)$ iff $(a, b)\cdot(-1, 2) = 0$, that is $a = 2b$. Therefore, any vector perpendicular to $(-1, 2)$ is of the form $(2b, b) = (2, 1)b$, where b is any real number.

   (b) Let (a, b, c) be any vector in $\mathbb{R}^3$, then (a, b, c) is perpendicular to $(2, 4, -1)$ iff $(a, b, c)\cdot(2, 4, -1) = 0$, that is, $c = 2a + 4b$. Therefore, any vector perpendicular to $(2, 4, -1)$ is of the form $(a, b, 2a + 4b) = (1, 0, 2)a + (0, 1, 4)b$, where a, b are any real numbers.

   (c) Let (a, b, c, d) be any vector in $\mathbb{R}^4$, then (a, b, c, d) is perpendicular to $(-1, 1, 1, -4)$ iff $(a, b, c, d)\cdot(-1, 1, 1, -4) = 0$, which gives $a = b + c - 4d$ Therefore, any vector perpendicular to $(-1, 1, 1, -4)$ is of the form $(b + c - 4d, b, c, d) = (1, 1, 0, 0)b + (1, 0, 1, 0)c + (-4, 0, 0, 1)d$, where b, c, d are any real numbers.

   These vectors are not unique. In general, all vectors perpendicular to a single one form a vector subspace. That is why there are some parameters in these answers. The number of parameters is equal to the dimension of this vector subspace, or degree of freedom so to speak.

3. Find all vectors that are perpendicular to

   (a) $(3, 2, 1), (1, 2, 3)$,

   (b) $(6, 0, -1), (2, 0, 0)$,

   (c) $(1, 2, 0, 0), (0, 1, -1, 0), (0, 1, 0, 1)$.

   Solution. (a) From $(3, 2, 1)\cdot(a, b, c) = 0,\ (1, 2, 3)\cdot(a, b, c) = 0$, we solve to get that these vectors are of the form $(a, b, c) = (1, -2, 1)a\ , a \in \mathbb{R}$.

   (b) They are $(0, 1, 0)b,\ b \in \mathbb{R}$.

   (c) They are $(2, -1, -1, 1)b\ ,\ b \in \mathbb{R}$.

4. Find all vectors that are perpendicular to the triangle formed by

(a) $(1,0,1),(1,0,-1),(0,1,1)$,

(b) $(-2,3,1),(4,0,1),(0,-5,1)$.

Solution. (a) $(a,b,c)\cdot((1,0,-1)-(1,0,1))=0,\ (a,b,c)\cdot((0,1,1)-(1,0,1))=0$ implies the vectors are $(1,-1,0)a,\ a\in\mathbb{R}$.

(b) They are $(0,0,1)c,\ c\in\mathbb{R}$.

5. Determine the angle between the vectors:

(a) $(-1,3),\ (6,3)$,

(b) $(1,-1,3)\ (-2,-1,-1)$,

(c) $(2,3,0,0),\ (7,-2,31,-109)$.

Solution.

(a) The angle θ is given by

$$\cos\theta=\frac{(-1,3)\cdot(6,3)}{|(-1,3)||(6,3)|}=\frac{-6+9}{\sqrt{10}\sqrt{45}}=\frac{3}{15\sqrt{2}}=\frac{1}{5\sqrt{2}}\ .$$

(b) The angle θ is given by

$$\cos\theta=\frac{(1,-1,3)\cdot(-2,-1,-1)}{|(1,-1,3)||(-2,-1,-1)|}=\frac{-2+1-3}{\sqrt{11}\sqrt{6}}=\frac{-4}{\sqrt{66}}\ .$$

(c) The angle θ is given by

$$\cos\theta=\frac{(2,3,0,0)\cdot(7,-2,31,-109)}{|(2,3,0,0)||(7,-2,31,-109)|}=\frac{14-6}{\sqrt{13}\sqrt{12895}}=\frac{8}{\sqrt{167635}}\ .$$

Note. I did not tailor problems to make the end results look good. We are lucky to have computer to do the job in practice.

6. (a) Let $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^n$. Show that the area of the triangle they form is given by

$$A=\frac{1}{2}\sqrt{|\mathbf{b}-\mathbf{a}|^2|\mathbf{c}-\mathbf{a}|^2-|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})|^2}\ .$$

(b) Find the area of the triangle with vertices at $(0,-1),(1,0),(-1,3)$.

(c) Find the area of the triangle with vertices at $(0,2,0),(1,2,3)$, and $(1,0,0)$.

Solution. (a). Let $\mathbf{u}=\mathbf{b}-\mathbf{a},\mathbf{v}=\mathbf{c}-\mathbf{a}$ and θ the angle between $\mathbf{u}$ and $\mathbf{v}$. The area is given by

$$\begin{aligned}A&=\frac{1}{2}|\mathbf{u}||\mathbf{v}|\sin\theta\\&=\frac{1}{2}|\mathbf{u}||\mathbf{v}|\sqrt{1-\cos^2\theta}\\&=\frac{1}{2}\sqrt{|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2}\ .\end{aligned}$$

(b) Take $\mathbf{u}=(1,1)$ and $\mathbf{v}=(-1,4)$ in the formula in (a) to get $A=5/2$.

(c) The area is $7/2$.

7. Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$. Verify that the vector

$$\mathbf{w} = \frac{|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}}{|\mathbf{u}| + |\mathbf{v}|} \ ,$$

bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution. Let θ be the angle between $\mathbf{u}$ and $\mathbf{w}$, and α be the angle between $\mathbf{v}$ and $\mathbf{w}$. We aim to show that $\cos\theta = \cos\alpha$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}||\mathbf{w}|} = \frac{\mathbf{u}\cdot(|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v})}{|\mathbf{u}||\mathbf{w}|(|\mathbf{u}| + |\mathbf{v}|)} = \frac{|\mathbf{u}|^2|\mathbf{v}| + |\mathbf{u}|(\mathbf{u}\cdot\mathbf{v})}{|\mathbf{u}||\mathbf{w}|(|\mathbf{u}| + |\mathbf{v}|)} = \frac{|\mathbf{u}||\mathbf{v}| + \mathbf{u}\cdot\mathbf{v}}{|\mathbf{w}|(|\mathbf{u}| + |\mathbf{v}|)} \ ,$$

and

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{\mathbf{v}\cdot(|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v})}{|\mathbf{v}||\mathbf{w}|(|\mathbf{u}| + |\mathbf{v}|)} = \frac{|\mathbf{u}||\mathbf{v}|^2 + |\mathbf{v}|(\mathbf{u}\cdot\mathbf{v})}{|\mathbf{v}||\mathbf{w}|(|\mathbf{u}| + |\mathbf{v}|)} = \frac{|\mathbf{u}||\mathbf{v}| + \mathbf{u}\cdot\mathbf{v}}{|\mathbf{w}|(|\mathbf{u}| + |\mathbf{v}|)} \ .$$

Therefore, $\cos\theta = \cos\alpha$, and hence $\mathbf{w}$ bisects $\mathbf{u}$ and $\mathbf{v}$. (Always keep in mind that the angle belongs to $[0, \pi]$ on which the cosine function is one-to-one.) Observe that $\mathbf{w}$ is of the form $a\mathbf{u} + b\mathbf{v}, a, b > 0$, $\mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

8. * Verify that the Euclidean distance satisfies the triangle inequality (the last axiom for a distance): For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} - \mathbf{y}| \le |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}| \ ,$$

and show that the equality sign in this inequality holds if and only if $\mathbf{z} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$ for some $t \in [0, 1]$. Hint: Use Cauchy-Schwarz Inequality.

Solution. Letting $\mathbf{a} = \mathbf{x} - \mathbf{z}, \mathbf{b} = \mathbf{z} - \mathbf{y}$ the inequality becomes

$$|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| \ .$$

Taking square,

$$(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \le |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2,$$

which is simplified to

$$\mathbf{a}\cdot\mathbf{b} \le |\mathbf{a}||\mathbf{b}| \ .$$

As $\mathbf{a}\cdot\mathbf{b} \le |\mathbf{a}\cdot\mathbf{b}|$, this inequality comes from Cauchy-Schwarz Inequality. For the equality case, it holds if and only if $\mathbf{a} = \lambda\mathbf{b}$ for some $\lambda > 0$ (in case $\mathbf{a}, \mathbf{b}$ are non-zero vectors). From $(\mathbf{x} - \mathbf{z}) = \lambda(\mathbf{z} - \mathbf{y})$ we get

$$\mathbf{z} = \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \ , \quad t = \frac{\lambda}{1 + \lambda} \in (0, 1) \ .$$

Clearly it also holds when one of $\mathbf{a}, \mathbf{b}$ is a zero vector.

9. *

   (a) Establish Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 \sum_{j=1}^{n} b_j^2 - \frac{1}{2}\sum_{j,k=1}^{n} \left(a_j b_k - a_k b_j\right)^2 \ .$$

(b) Deduce Cauchy-Schwarz Inequality from it.

Solution. (a) We play with the indices:

$$\begin{aligned}
\Big(\sum_j a_j^2\Big)\Big(\sum_k b_k^2\Big) - \Big(\sum_k a_k b_k\Big)^2 &= \Big(\sum_j a_j^2\Big)\Big(\sum_k b_k^2\Big) - \Big(\sum_k a_k b_k \sum_j a_j b_j\Big) \\
&= \frac{1}{2}\sum_{j,k}(a_j^2 b_k^2 + a_k^2 b_j^2) - \Big(\sum_k a_k b_k\Big)\Big(\sum_j a_j b_j\Big) \\
&= \frac{1}{2}\sum_{j,k}\left(a_j^2 b_k^2 - 2a_j b_k a_k b_j + a_k^2 b_j^2\right) \\
&= \frac{1}{2}\sum_{j,k}(a_j b_k - a_k b_j)^2 \ .
\end{aligned}$$

(b) The inequality follows immediately from this identity since its RHS is always non-negative. With further work, one can show that $a_i b_j - a_j b_i = 0$ for all $i < j$ implies that $\mathbf{b} = \alpha\mathbf{a}$ for some α when they are non-zero vectors.

10. * Prove the polarization identity in $\mathbb{R}^n$:

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{4}\left(|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\right) \ .$$

What is its meaning?

Solution. It follows from adding up

$$|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \ ,$$

and

$$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \ .$$

The meaning of this identity is that the angle between $\mathbf{a}, \mathbf{b}$ can be recovered from the norms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \pm \mathbf{b}$.

11. * For $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$, define

$$\|x - y\| = |x_1 - y_1| + |x_2 - y_2| \ .$$

(a) Show that $\|\mathbf{x} - \mathbf{y}\|$ defines a distance on $\mathbb{R}^2$.

(b) Show that there are points on the circle $x_1^2 + x_2^2 = 1$ whose distances to the origin as defined in (a) are different.

(c) Draw the unit circle $\{(x_1, x_2) : \|(x_1, x_2)\| = 1\}$.

Solution.

(a) This is done by checking the three conditions in the definition of a metric.

(b) Consider $P = (1, 0), Q = (1/\sqrt{2}, 1/\sqrt{2})$, which are points on the circle. With respect to the above distance, we have
$\|P - (0, 0)\| = 1 + 0 = 1$, and
$\|Q - (0, 0)\| = \sqrt{2}$,
which are different.

(c) The unit circle with respect to this metric is the square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

12. Describe the Euclidean motion

$$T\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} ,$$

in terms of translation, rotation and reflection.

Solution. Since $\cos(-\pi/4) = 1/\sqrt{2}, \sin(-\pi/4) = -1\sqrt{2}$, we may write

$$Tx = \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Therefore, T is obtained by first a clockwise rotation by $\pi/4$, and then follows by a translation by $(-3, 2)$.

13. Find the Euclidean motion that fixes the origin and send $(-1, 1)$ to $(\sqrt{2}, 0)$.

Solution. Let T be the required Euclidean motion with matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then A satisfies

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} .$$

In other words, we have the following system of linear equations

$$\begin{cases} -\cos\theta - \sin\theta & = \sqrt{2} , \\ -\sin\theta + \cos\theta & = 0 , \end{cases}$$

which is readily solved to get $(\cos\theta, \sin\theta) = (-\sqrt{2}/2, -\sqrt{2}/2,)$, and hence

$$A = \frac{\sqrt{2}}{2} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

is the matrix of the required Euclidean motion.

14. Find the formula for the reflection with respect to the y-axis.

Solution. Under this reflection (x, y) is mapped to $(-x, y)$, that is, $(1, 0) \mapsto (-1, 0)$ and $(0, 1) \mapsto (0, 1)$. The Euclidean motion is given by $T(x, y) = A(x, y)$ where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

15. Find the formula for the reflection with respect to the straight line $x - 2y = 0$.

Solution. Such a formula could be obtained by composing the following rigid motions:

(1) rotate clockwise by an angle θ such that the line $x - 2y = 0$ is transformed to the x-axis (with equation $y = 0$).

(2) reflect along x-axis

(3) rotate anti-clockwise by the same angle θ as in (1).

To obtain an explicit formula, it suffices to find the matrices correspond to each of the above rigid motions:

(1) : consider the vector $(2,1)$ on the line. The angle θ is exactly the angle between $(2,1)$ and the positive real axis.Therefore, $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$. Since this is a clockwise motion, we should take $-\theta$ instead, and hence $\cos(-\theta) = \frac{2}{\sqrt{5}}$ and $\sin(-\theta) = -\frac{1}{\sqrt{5}}$.

Therefore, the corresponding matrix is $A_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

(2) Clearly the matrix correspond to reflection is $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(3) Similar argument as in (1), using θ instead of $-\theta$, we have the corresponding matrix is $A_3 = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

Therefore, the matrix corresponding to the given reflection is $A_3A_2A_1$ which is given by

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} .$$

16. Determine the magnitude and direction of the following cross product:
    (a) $(0,-2,-9) \times (1,2,-4)$,
    (b) $(0,-6,1) \times (3,0,5)$,
    (c) $(1,-1,2) \times (-2,2,-4)$.

    Solution.

    (a) $(0,-2,-9) \times (1,2,-4) = (26,-9,2)$. Therefore, magnitude $= \sqrt{26^2+(-9)^2+2^2} = \sqrt{761}$, direction $= \frac{1}{\sqrt{761}}(26,-9,2)$
    (b) $(0,-6,1) \times (3,0,5) = (-30,3,18)$. Therefore, magnitude is equal to
    $$\sqrt{(-30)^2+3^2+18^2} = \sqrt{1233} ,$$
    and direction
    $$\frac{1}{\sqrt{1233}}(-30,3,18) .$$
    (c) $(1,-1,2) \times (-2,2,-4) = (0,0,0)$. Therefore, its magnitude is equal to 0 and its direction is undefined.

17. Consider the points $(1,1,0), (1,0,1)$ and $(0,1,1)$.
    (a) Find the area of the triangle with vertices at these points.
    (b) Find the volume of the parallelepiped with vertices at these points together with $(0,0,0)$.

    Solution.

    (a) First we translate the triangle so that $(0,1,1)$ goes to the origin. Let $\mathbf{u} = (1,1,0) - (0,1,1) = (1,0,-1), \mathbf{v} = (1,0,1) - (0,1,1) = (1,-1,0)$. Then $\mathbf{u}$ and $\mathbf{v}$ are two sides of the triangle. The area of the triangle is equal to
    $$\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}|(-1,-1,-1)| = \frac{\sqrt{3}}{2} .$$

(b) Its volume:

$$|(1,1,0)\cdot((1,0,1)\times(0,1,1))| = |(1,1,0)\cdot(-1,-1,1)| = 2.$$

18. Consider $(1,3,-2),(2,4,5)$ and $(-3,-2,2)$.

(a) Find the area of the triangle with vertices at these points.

(b) Find the volume of the parallelepiped with vertices at these points together with $(0,0,0)$.

Solution.

(a) Let $\mathbf{u} = (1,3,-2)-(-3,-2,2) = (4,5,-4), \mathbf{v} = (2,4,5)-(-3,-2,2) = (5,6,3)$. Then $\mathbf{u}$ and $\mathbf{v}$ are two sides of the triangle. Its area is given by

$$\frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}|(39,-32,-1)| = \frac{\sqrt{2546}}{2} \ .$$

(b) Its volume:

$$|(2,4,5)\cdot((1,3,-2)\times(-3,-2,2))| = |(2,4,5)\cdot(2,4,7)| = 55 \ .$$

19. Determine whether the following points are coplanar or not:

$$(1,3,-2),\ (3,4,1),\ (2,0,-2),\ (4,8,4) \ .$$

Solution. Let $\mathbf{u} = (1,3,-2)-(4,8,4) = (-3,-5,-6), \mathbf{v} = (3,4,1)-(4,8,4) = (-1,-4,-3), \mathbf{w} = (2,0,-2)-(4,8,4) = (-2,-8,-6)$. Using $\mathbf{w} = 2\mathbf{v}$, the volume of the parallelepiped formed by $\mathbf{u},\mathbf{v},\mathbf{w}$ is given by $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = |\mathbf{u}\cdot 0| = 0$. Therefore, they are coplanar.

20. (a) Establish

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \ .$$

(b) Show that

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) \ .$$

Solution. (a) follows from the determinant formula for the cross product and (b) follows from (a) and the properties of the determinant.

21. (a) Let $\mathbf{u} = (u_1,u_2,u_3)$ and $\mathbf{v} = (v_1,v_2,v_3)$ be two non-zero vectors satisfying

$$u_2v_3 - u_3v_2 = 0,\ u_1v_3 - u_3v_1 = 0,\ u_1v_2 - u_2v_1 = 0 \ .$$

Show that there is some non-zero α such that $\mathbf{v} = \alpha\mathbf{u}$.

(b) Show that two non-zero vectors in $\mathbb{R}^3$ satisfying $\mathbf{u}\times\mathbf{v} = 0$ if and if $\mathbf{v} = \alpha\mathbf{u}$ for some non-zero α.

Solution.

(a) Assume $\mathbf{u}$ is a non-zero vector. Without loss of generality, let $u_1 \neq 0$. Set $\alpha = \dfrac{v_1}{u_1}$. Then $v_1 = \alpha u_1$, and hence by second equation we have $u_1(v_3 - \alpha u_3) = 0$. Since $u_1 \neq 0, v_3 - \alpha u_3 = 0$; similarly, using the third equation we have $v_2 - \alpha u_2 = 0$. Altogether we have $\mathbf{v} = \alpha\mathbf{u}$, where $\alpha \neq 0$ for otherwise $\mathbf{v} = \mathbf{0}$.

(b) ($\Rightarrow$) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, by definition of cross product, we have

$$u_2v_3 - u_3v_2 = 0,\ u_1v_3 - u_3v_1 = 0,\ u_1v_2 - u_2v_1 = 0\ .$$

and hence the result follows from (a).
($\Leftarrow$) If $\mathbf{v} = \alpha\mathbf{u}$, then $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times (\alpha\mathbf{u}) = \alpha(\mathbf{u} \times \mathbf{u}) = 0$

22. * Verify the following properties for the cross product: For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and scalars α, β,

(a)
$$\alpha(\mathbf{u} \times \mathbf{v}) = (\alpha\mathbf{u}) \times \mathbf{v}\ ,$$

(b)
$$\mathbf{u} \times (\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\mathbf{u} \times \mathbf{v} + \beta\mathbf{u} \times \mathbf{w}\ .$$

Solution.

(a) $\alpha(\mathbf{u} \times \mathbf{v}) = \alpha(u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)$
$= ((\alpha u_2)v_3 - (\alpha u_3)v_2, -((\alpha u_1)v_3 - (\alpha u_3)v_1), (\alpha u_1)v_2 - (\alpha u_2)v_1) = (\alpha u) \times v$

(b) $\mathbf{u} \times (\alpha\mathbf{v} + \beta\mathbf{w}) = \mathbf{u} \times (\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2, \alpha v_3 + \beta w_3)$
$= (u_2(\alpha v_3 + \beta w_3) - u_3(\alpha v_2 + \beta w_2), -(u_1(\alpha v_3 + \beta w_3) - u_3(\alpha v_1 + \beta w_1), u_1(\alpha v_2 + \beta w_2) - u_2(\alpha v_1 + \beta w_1))$
$= ((\alpha u_2)v_3 - (\alpha u_3)v_2, -((\alpha u_1)v_3 - (\alpha u_3)v_1), (\alpha u_1)v_2 - (\alpha u_2)v_1)$
$+ ((\beta u_2)w_3 - (\beta u_3)w_2, -((\beta u_1)w_3 - (\beta u_3)w_1), (\beta u_1)w_2 - (\beta u_2)w_1))$
$= \alpha\mathbf{u} \times \mathbf{v} + \beta\mathbf{u} \times \mathbf{w}\ .$

23. * For 3-vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, establish the following identities:

(a)
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})\ .$$

(b) (Jacobi's identity)
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0\ .$$

Solution. (b) comes from (a). To prove (a), we have

$$\begin{aligned}(\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_1 &= u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3)\\ &= v_1(u_2w_2 + u_3w_3) - w_1(u_2v_2 + u_3v_3)\\ &= v_1(\mathbf{u} \cdot \mathbf{w}) - w_1(\mathbf{u} \cdot \mathbf{v})\ .\end{aligned}$$

And similar formulas hold for the other two components.

24. Let P, Q, and R be three points in space. Denote by $\overline{PQ}$ the vector from Q to P etc. Show that the distance from R to $\overline{PQ}$ is given by the formula

$$d = \frac{|\overline{PR} \times \overline{QR}|}{|\overline{PQ}|}\ .$$

Solution. If R lies on the straight line passing P, Q, then either $\overline{PR} = \alpha\overline{QR}$ or $\overline{QR} = \alpha\overline{PR}$ for some α (depending on whether R coincide with P or Q). In either case, we have $\overline{PR} \times \overline{QR} = 0$, and hence $d = 0$ as expected since P, Q, R are collinear.

If R does not lie on the straight line passing P, Q, then P, Q, R determines a triangle in space. Computing its area a in 2 ways:

(i) $a = \frac{1}{2}|\overline{PQ}|d$

(ii) $a = \frac{1}{2}\left|\overline{PR}\right|\left|\overline{QR}\right|\sin\angle PRQ = \frac{1}{2}\left|\overline{PR}\times\overline{QR}\right|$

Therefore, $\frac{1}{2}|\overline{PQ}|\ d = \frac{1}{2}\left|\overline{PR}\times\overline{QR}\right|$, and hence

$$d = \frac{\left|\overline{PR}\times\overline{QR}\right|}{|\overline{PQ}|}\ .$$

25. * Let P, Q, R be three points lying in a plane in $\mathbb{R}^3$ and S another point. Show that the distance from S to the plane is given by

$$d = \frac{\left|\overline{SP}\cdot(\overline{SQ}\times\overline{SR})\right|}{|\overline{QP}\times\overline{RP}|}\ .$$

Suggestion: Look at the volume of the pyramid formed by these four points. The volume of a pyramid is given by $\frac{1}{3}ha$ where h is its height and a its base area. It is equal to $\frac{1}{6}$ of the parallelepiped spanned by these vectors (taking S as the origin).

Solution.

If S lies on the plane determined by P, Q, R, then $\overline{SP}\cdot(\overline{SQ}\times\overline{SR}) = 0$, and hence $h = 0$ as expected.

If S does not lie on the plane determined by P, Q, R, following the hint and compute the volume V of the pyramid determined by these four points in two ways:

(i) $V = \frac{1}{6}da = \frac{1}{6}d|\overline{QP}\times\overline{RP}|$

(ii) $V = \frac{1}{6}$ (volume of the parallelepiped determined by $\overline{SP}, \overline{SQ}, \overline{SR}$) $= \frac{1}{6}\left|\overline{SP}\cdot(\overline{SQ}\times\overline{SR})\right|$

Therefore, $\frac{1}{6}d\cdot|\overline{QP}\times\overline{RP}| = \frac{1}{6}\left|\overline{SP}\cdot(\overline{SQ}\times\overline{SR})\right|$, and hence

$$d = \frac{\left|\overline{SP}\cdot(\overline{SQ}\times\overline{SR})\right|}{|\overline{QP}\times\overline{RP}|}\ .$$

26. * Show that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by half of the absolute value of the determinant, that is,

$$\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}\ .$$

Solution. Consider the triangle in space formed by $(1, x_1, y_1), (1, x_2, y_2), (1, x_3, y_3)$. Using the first point as the base point, the signed area of the triangle is given by 1/2 of $(0, x_2 - x_1, y_2 - y_1)\times(0, x_3 - x_1, y_3 - y_1)$ which is equal to

$$\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}\ .$$